

A PROPOSED STANDARD SET OF PROBLEMS TO TEST FINITE ELEMENT ACCURACY

Richard H. MACNEAL and Robert L. HARDER

The MacNeal–Schwendler Corporation, 815 Colorado Boulevard, Los Angeles, CA 90041, U.S.A.

Received February 1984

Abstract. A proposed standard set of test problems is described and applied to representative quadrilateral plate and solid brick finite elements. The problem set contains patch tests and beam, plate, and shell problems, some of which have become de facto standards for comparing the accuracy of finite elements. Although few in number, the tests are able to display most of the parameters which affect finite element accuracy.

Introduction

The intended purpose of the proposed problem set is to help users and developers of finite element programs to ascertain the accuracy of particular finite elements in various applications. It is not intended that the problems be used as benchmarks for cost comparisons since the problems are, in general, too small to be meaningful for this purpose.

Nothing is as important to the success of a finite element analysis as the accuracy of the elements. Indeed, in a linear static analysis, the finite elements embody all of the discretizing assumptions; the rest of the calculations are exact except for their lack of precision. Thus the accuracy of the finite elements should be a matter of primary concern to those who perform finite element analyses and to those who are responsible for conclusions derived therefrom.

Every new finite element is tested with one or more small problems and the results of these tests are generally published in the open literature or in FEM program documentation. Such test results are all that the user has to help him evaluate the elements prior to actual use. They have invariably proven to be woefully inadequate for this purpose: (a) because they test an insufficient number of conditions, (b) because few, if any, bad results are reported (not by design but because the developer fixed only the bugs he discovered), and (c) because they usually cannot be compared with results for other elements, particularly with those in other programs. These same defects would not be present in a carefully designed set of standard test problems applied to many different finite elements and widely circulated.

The need for verifying finite element accuracy by independent testing and for comparing finite element results is becoming more widely recognized. A recent effort to evaluate the plate elements in commercial programs [1,2] revealed results ranging from excellent to extremely poor and misleading. Governmental concern for the accuracy of finite element analysis is evidenced by the Nuclear Regulatory Commission's requirement for structural analysis computer program validation, and abroad by the recent formation in the United Kingdom of a National Agency for Finite Element Methods and Standards (NAFEMS). While the former agency relies on verification problems supplied by the developer, the latter shows promise of doing their own testing.

The authors can confirm from personal experience that the design of an adequate set of problems for finite element testing requires careful planning. Too often initial testing is done

with one or two relatively easy problems with known answers. For example, the only problem used in development testing of the NASTRAN[®] plate bending elements TRPLT and QDPLT was the lateral loading of a rectangular plate [3]. The subsequent use of QDPLT and its membrane counterpart QDMEM in the solution of verification problems did not reveal any weaknesses in these elements because the verification problems were chosen to give excellent comparisons with theory rather than to act as critical element tests. We will show in this paper that these two elements (which when combined form the QUAD2 element) in fact have many weaknesses. This example is not an isolated one. Every element in MSC/NASTRAN has been revised in response to difficulties encountered in the field. In that process, we have gradually built up a library of element test problems which clearly demonstrate frequently encountered element failure modes. The problems to be described in this paper represent a substantial part of that library.

The most important symptoms of accuracy failure in modern finite elements are spurious mechanisms, also known as rank deficiencies, and a phenomenon known as locking in which excessive stiffness is exhibited for particular loadings and/or irregular shapes. Most elements display one or the other of these symptoms, but not usually both. An important state-of-the-art problem is the design of elements which are free from spurious mechanisms and locking in *all* situations. Elementary defects of element design, such as violation of rigid body property and noninvariance to node numbering, are less frequently encountered nowadays, but are devastating when they occur.

The design of a comprehensive set of element test problems should, of course, take into account the parameters which affect accuracy. These parameters can be classified under the headings of loading, element geometry, problem geometry, and material properties.

With regard to loading, the problem set should, as a whole, provide significant loading for

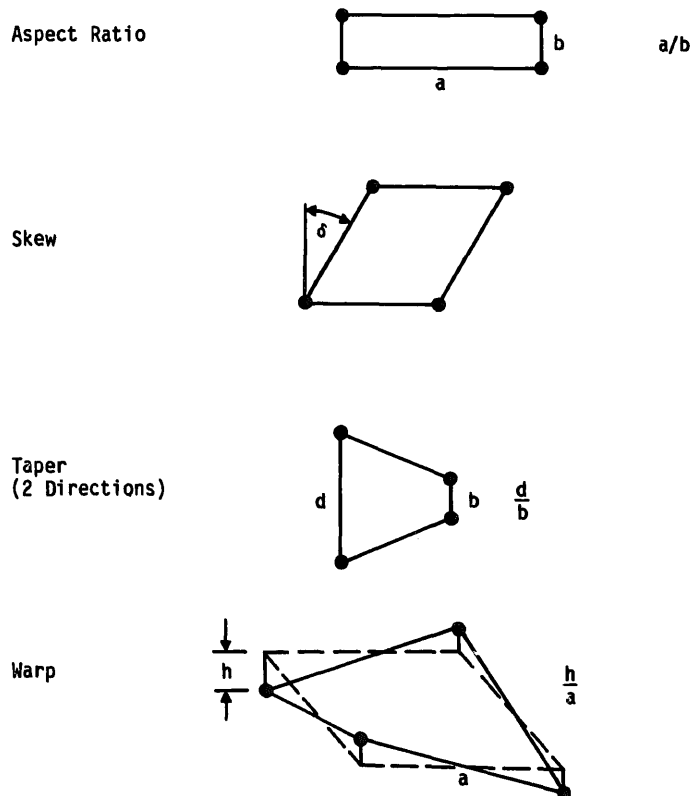


Fig. 1. Types of geometric distortion from a square plate.

each of the types of deformation which the elements can exhibit. For example, a three-noded shell element should be subjected to extension, in-plane shear, and out-of-plane bending. For a four-noded shell element, add in-plane bending and twist. For an eight-noded element, add the motion of the edge nodes relative to the corners. The latter are less important, but should not be neglected entirely.

Each element has a standard shape which may be the only shape that the developer has tested. In the case of a quadrilateral, the standard shape is a square; in the case of a hexahedron, the standard shape is a cube; and in the case of a triangle, the standard shape is usually an isosceles right triangle. Care should be taken to test nonstandard shapes. Fig. 1 shows the four basic modes of distortion of a square, each of which should be exhibited in the test problems and tested with several kinds of loading.

Geometric parameters which are not isolated to single elements can also affect element accuracy. Curvature is the most important such parameter. It is not sufficient to test only single curvature since some elements which behave well for single curvature behave poorly for double curvature. The slenderness ratio and the manner of support of a structure affect the conditioning of the stiffness matrix and therefore can be used to check element failures related to precision.

Poisson's ratio has a strong effect on element accuracy as its value approaches 0.5. Such values should be included in the problem set if the use of nearly incompressible materials is contemplated. Plasticity affects element accuracy in much the same way as incompressible material. Plasticity and all other nonlinear effects are outside the scope of the paper. Anisotropic material properties also have a significant effect on element accuracy which will not be examined here.

The test problems

The names of the proposed test problems are listed in Table 1, which also indicates the suitability of each problem for testing various types of elements. The geometry, material properties, boundary conditions, loading, and element meshing for each problem are described in Figs. 2 through 10 in sufficient detail to permit construction of a finite element model

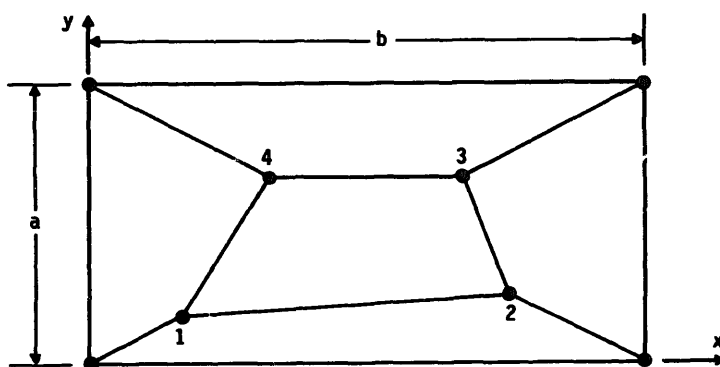


Fig. 2. Patch test for plates. $a = 0.12$; $b = 0.24$; $t = 0.001$; $E = 1.0 \times 10^6$; $\nu = 0.25$. Boundary conditions: see Table 2. Location of inner nodes:

	x	y
1	0.04	0.02
2	0.18	0.03
3	0.16	0.08
4	0.08	0.08

Table 1
Summary of proposed test problems

Test problem	Suitability of problem for element type				
	Beam	Membrane plate	Bending plate	Shell ^a	Solid
Patch tests		×	×		×
Straight cantilever beam	×	×	×		×
Curved beam	×	×	×		×
Twisted beam	×			×	×
Rectangular plate			×		×
Scordelis-Lo roof				×	×
Spherical shell				×	×
Thick-walled cylinder		×			×

^a A shell element is defined here as an element that combines membrane and bending properties.

^b Using plane strain option.

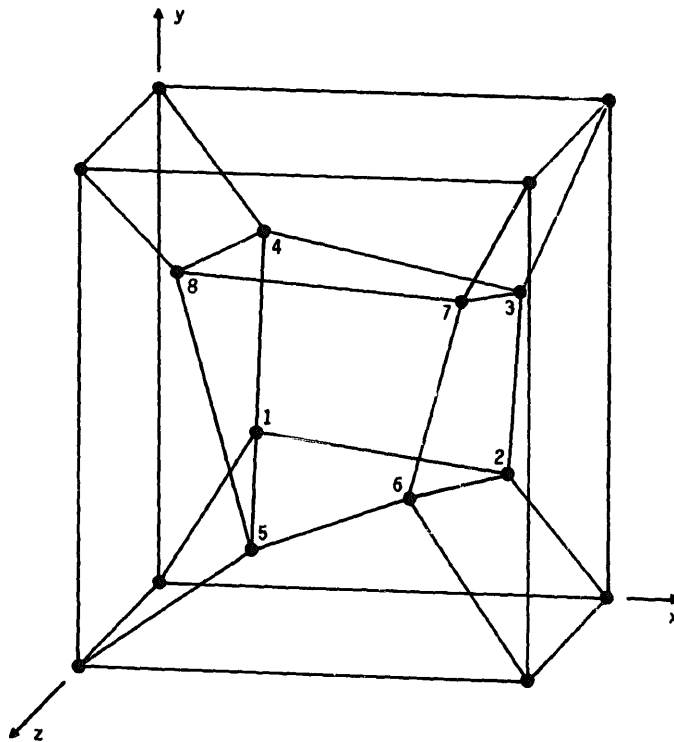


Fig. 3. Patch test for solids. Outer dimensions: unit cube; $E = 1.0 \times 10^6$; $\nu = 0.25$. Boundary conditions: see Table 2. Location of inner nodes:

	x	y	z
1	0.249	0.342	0.192
2	0.826	0.288	0.288
3	0.850	0.649	0.263
4	0.273	0.750	0.230
5	0.320	0.186	0.643
6	0.677	0.305	0.683
7	0.788	0.693	0.644
8	0.165	0.745	0.702

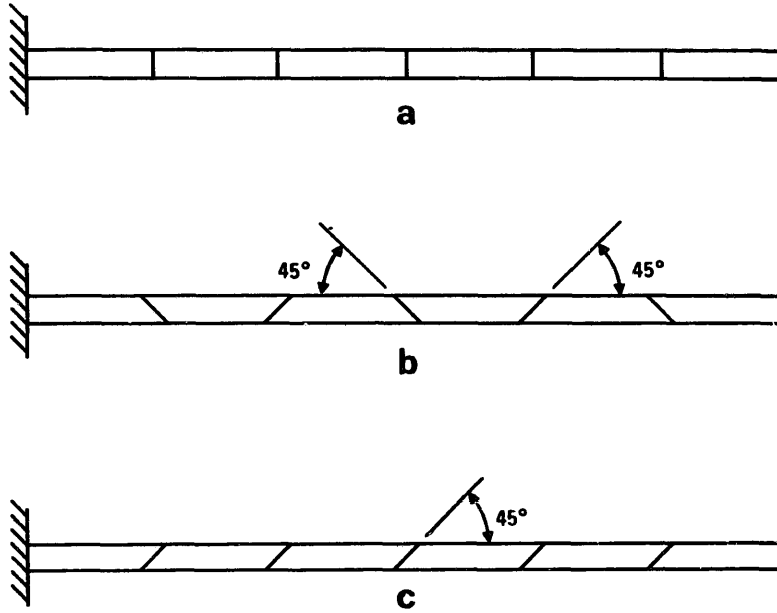


Fig. 4. Straight cantilever beam. (a) Regular shape elements; (b) Trapezoidal shape elements; (c) Parallelogram shape elements. Length = 6.0; width = 0.2; depth = 0.1; $E = 1.0 \times 10^7$; $\nu = 0.30$; mesh = 6×1 . Loading: unit forces at free end. (Note: All elements have equal volume.)

consisting of beam, quadrilateral plate, shell, or brick elements. An appropriate meshing for triangles and wedge elements can be obtained by subdividing the quads and bricks. Theoretical results for the problems are given in Tables 2 through 5.

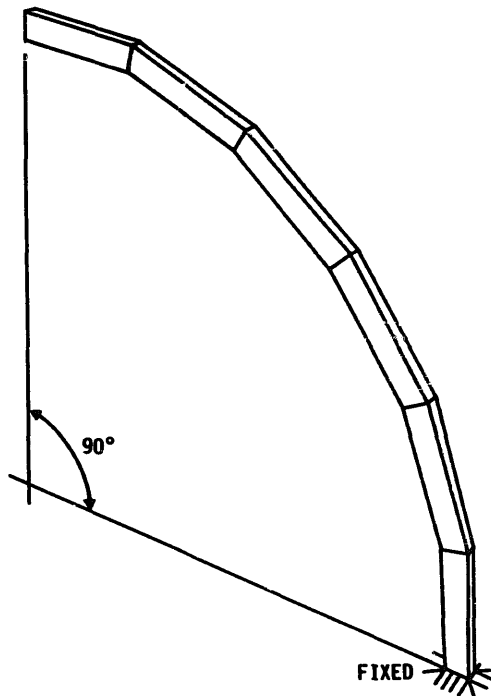


Fig. 5. Curved beam. Inner radius = 4.12; outer radius = 4.32; arc = 90° ; thickness = 0.1; $E = 1.0 \times 10^7$; $\nu = 0.25$; mesh = 6×1 . Loading: unit forces at tip.

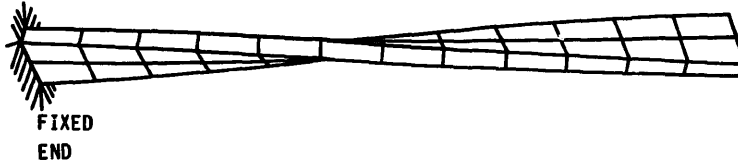


Fig. 6. Twisted beam. Length = 12.0; width = 1.1; depth = 0.32; twist = 90° (root to tip); $E = 29.0 \times 10^6$; $\nu = 0.22$; mesh = 12×2 . Loading: unit forces at tip.

No comprehensive set of finite element test problems would be complete if it did not include patch tests for plate and solid problems. The patch test that we propose for plates, shown in Fig. 2, has been used by Robinson [1,2] to test commercial finite elements. Note that the arbitrarily distorted element shapes are an essential part of the test. On the other hand, the rectangular exterior shape of the plate makes it easy to provide boundary conditions corresponding to constant membrane strains or constant bending curvatures, independent of element shape. We have elected to use displacement boundary conditions (see Table 2) because they are easier to specify for a variety of elements than the force and moment boundary conditions

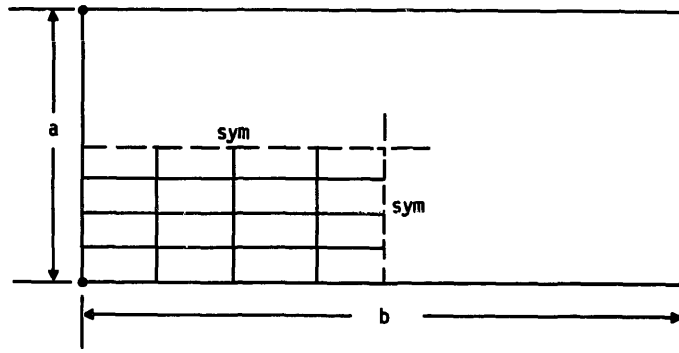


Fig. 7. Rectangular plate. $a = 2.0$; $b = 2.0$ or 10.0 ; thickness = 0.0001 (plates); thickness = 0.01 (solids); $E = 1.7472 \times 10^7$; $\nu = 0.3$; boundaries = simply supported or clamped; mesh = $N \times N$ (on $1/4$ of plate). Loading: uniform pressure, $q = 10^{-4}$, or central load $P = 4.0 \times 10^{-4}$.

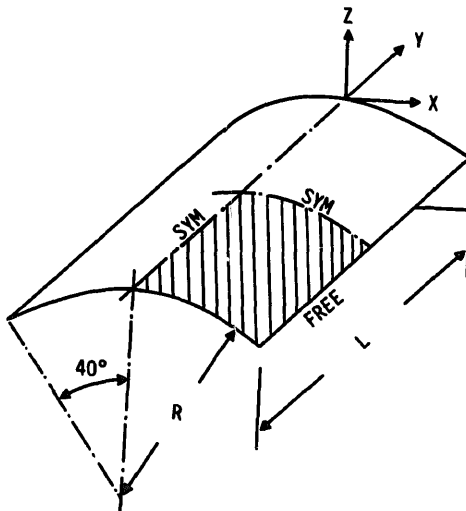


Fig. 8. Scordelis-Lo roof. Radius = 25.0; length = 50.0; thickness = 0.25; $E = 4.32 \times 10^8$; $\nu = 0.0$; loading = 90.0 per unit area in $-Z$ direction; $u_x = u_z = 0$ on curved edges; mesh: $N \times N$ on shaded area.

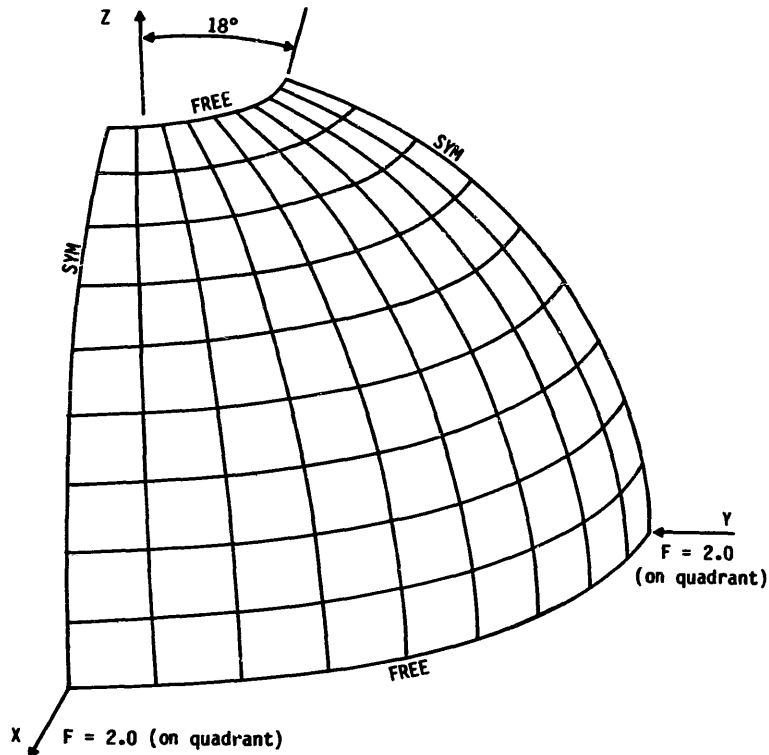


Fig. 9. Spherical shell problem. Radius = 10.0; thickness = 0.04; $E = 6.825 \times 10^7$; $\nu = 0.3$; mesh = $N \times N$ (on quadrant). Loading: concentrated forces as shown.

employed by Robinson. If the latter are used, the load distribution rules shown in Fig. 11 are appropriate for isoparametric elements.

The principal virtue of a patch test is that, if an element produces correct results for the test, the results for any problem solved with the element will converge toward the correct solution as

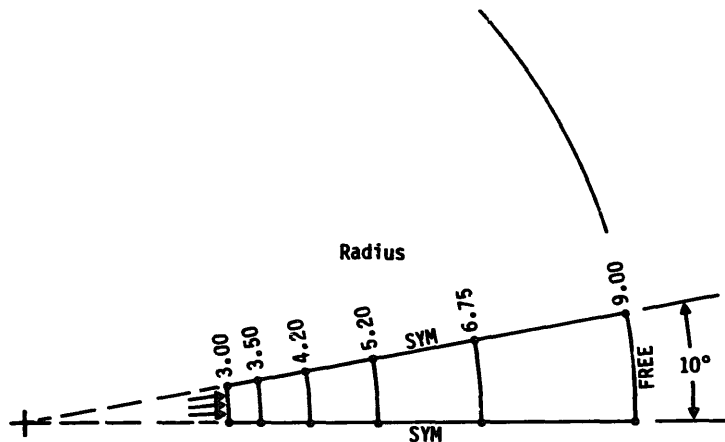


Fig. 10. Thick-walled cylinder. Inner radius = 3.0; outer radius = 9.0; thickness = 1.0; $E = 1000$; $\nu = 0.49, 0.499, 0.4999$; plane strain condition; mesh: 5×1 (as shown above). Loading: unit pressure at inner radius.

Table 2
Boundary conditions and theoretical solutions for patch tests

(a) Membrane plate patch test	
Boundary conditions: $u = 10^{-3}(x + y/2)$ $v = 10^{-3}(y + x/2)$	
Theoretical solution: $\epsilon_x = \epsilon_y = \gamma = 10^{-3}$; $\sigma_x = \sigma_y = 1333$; $\tau_{xy} = 400$	
(b) Bending plate patch test	
Boundary conditions: $w = 10^{-3}(x^2 + xy + y^2)/2$ $\theta_x = \partial w / \partial y = 10^{-3}(y + x/2)$ $\theta_y = -\partial w / \partial x = 10^{-3}(-x - y/2)$	
Theoretical solution: Bending moments per unit length: $m_x = m_y = 1.111 \times 10^{-7}$; $m_{xy} = 10^{-7}$ Surface stresses: $\sigma_x = \sigma_y = \pm 0.667$; $\tau_{xy} = \pm 0.200$	
(c) Solid patch test	
Boundary conditions: $u = 10^{-3}(2x + y + z)/2$ $v = 10^{-3}(x + 2y + z)/2$ $w = 10^{-3}(x + y + 2z)/2$	
Theoretical solution: $\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 10^{-3}$ $\sigma_x = \sigma_y = \sigma_z = 2000$; $\tau_{xy} = \tau_{yz} = \tau_{zx} = 400$	

Table 3
Theoretical solutions for beam problems

Tip load direction	Displacement in direction of load		
	Straight beam	Curved beam	Twisted beam
Extension	3.0×10^{-5}	—	—
In-plane shear	0.1081	0.08734	0.005424
Out-of-plane shear	0.4321	0.5022	0.001754
Twist	0.03208	—	—

Table 4
Theoretical solutions for rectangular plate ^a

Boundary supports	Aspect ratio b/a	Displacement at center of plate ^b	
		Uniform pressure	Concentrated force
Simple	1.0	4.062	11.60
Simple	5.0	12.97	16.96
Clamped	1.0	1.26	5.60
Clamped	5.0	2.56	7.23

^a See [8] for the method.

^b Values shown are for shell elements; multiply by 10^{-6} for solid elements because the thickness used for solid elements is 100 times greater. Erratic results were obtained with the solid elements at the lesser thickness.

Table 5
Theoretical solutions for shell and cylinder problems

(a) *Scordelis–Lo roof*

The value for the midside vertical displacement quoted in [5] is 0.3086. Many finite elements converge to a slightly smaller value. We have used the value 0.3024 for normalization of our results.

(b) *Spherical shell*

Reference [7] computes a theoretical lower bound of 0.0924 for the displacement under load in the case where the hole at the center is not present. We have used the value of 0.0940 for normalization of our results.

(c) *Thick-walled cylinder*

Formula for radial displacement:

$$u = \frac{(1 + \nu) p R_1^2}{E(R_2^2 - R_1^2)} \left[R_2^2/r + (1 - 2\nu)r \right]$$

where p = pressure; R_1 = inner radius; R_2 = outer radius.

Poisson's ratio	Radial displacement at $r = R_1$
0.49	5.0399×10^{-3}
0.499	5.0602×10^{-3}
0.4999	5.0623×10^{-3}

the elements are subdivided. The reason, of course, is that the stress within each element tends to a uniform value in the limit. Many authorities, including Irons [4], feel that an element that does not pass the patch test should not be trusted. On the other hand, passing the patch test does not guarantee satisfaction since the rate of convergence may be too slow for practical use.

The proposed patch test for solid elements (see Fig. 3) is seen to be an extension of Robinson's patch test to three dimensions. Displacement boundary conditions for the test are also given in Table 2.

The straight cantilever beam (see Fig. 4) is a frequently used test problem which can be applied to beam, plate, and solid elements.¹ Its virtues are its simplicity and the fact that all of the principal element deformation modes described earlier (constant and linearly varying strains and curvatures) can be evoked by loads applied to the free end. We have added irregular element shapes principally to test the combination of such shapes with linearly varying strains.

In the curved cantilever beam (see Fig. 5), combinations of the principal deformation modes are evoked by a single in-plane or out-of-plane shear load at the tip. Note also that the element shape is not quite rectangular, which will test the effect of slight irregularity. Theoretical tip displacement results for all of the beam problems are shown in Table 3.

The twisted beam element is the only one in our proposed set that tests the effect of warp on plate elements. The warp of each element is only 7.5° , but even this small amount will produce a surprising result.

As noted earlier, the laterally loaded rectangular plate (see Fig. 7) is a survivor from the original testing of NASTRAN® plate elements. It has become a de facto standard test and has been seen frequently in the technical literature. Theoretical results for lateral displacement at the center are provided in Table 4 for all of the combinations of boundary supports, aspect ratio, and loading conditions listed in Fig. 7. This is the first problem in which convergence with decreasing mesh spacing will be studied.

¹ Note that the proper way to apply free end load depends on element type (see Fig. 11).

The Scordelis–Lo roof [5] (see Fig. 8) also has achieved the status of a de facto standard test problem and has found its way into textbooks [6]. The test result most frequently displayed is the vertical displacement at the midpoint of the free edge. The theoretical value for this result is 0.3086, but most elements converge to a slightly lower value. Membrane and bending deformations both contribute significantly to it. The Scordelis–Lo roof is the only singly-curved shell problem in the proposed standard problem set.

The spherical shell shown in Fig. 9 is our proposed doubly-curved shell problem. Note that the equator is a free edge so that the problem represents a hemisphere with four point loads alternating in sign at 90° intervals on the equator. The hole at the top has been introduced to avoid the use of triangles near the axis of revolution. Convergence can be studied by varying mesh size. Both membrane and bending strains contribute significantly to the radial displacement at the load point. A theoretical value of the displacement under load has been computed for a slightly different configuration [7] in which the hole at the axis is closed.

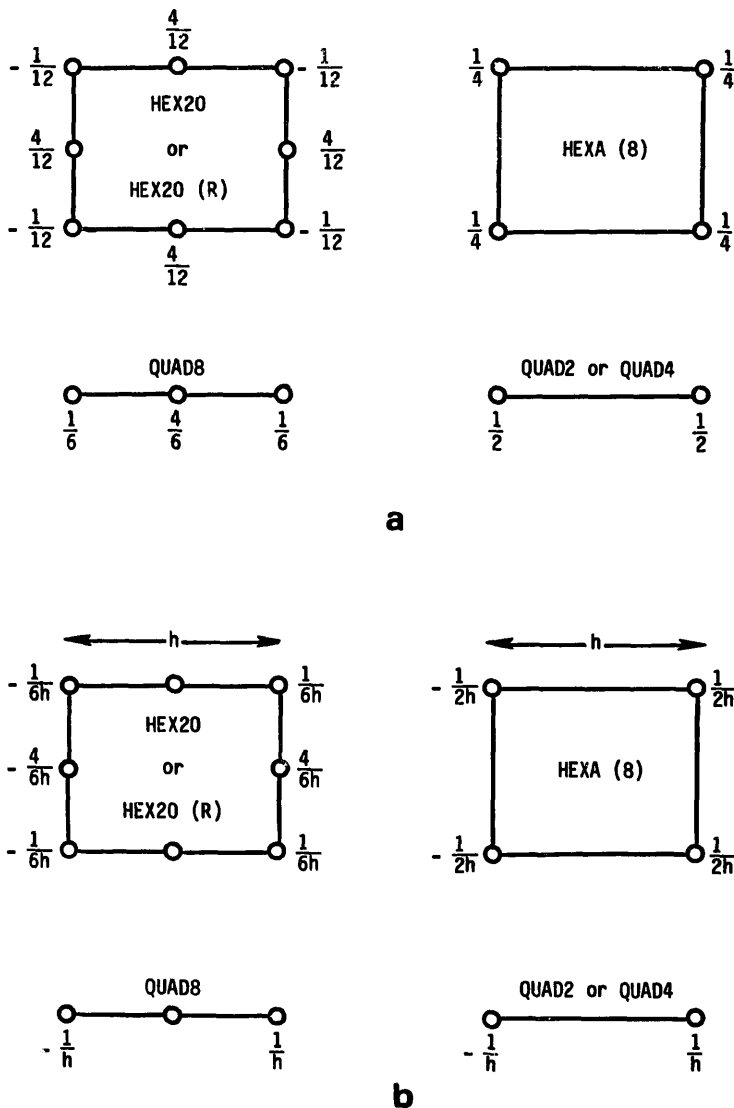








Fig. 11. Consistent load distribution on beam end. (a) Tension or shear load. (b) Moment load.

Table 6
Summary of element properties

Name	Shape	Connections		Description	Reference
		Number of points	Components ^a		
QUAD2		4	<i>T, R</i>	Composed of two pairs of overlapping triangles. Each triangle has constant membrane strain and lateral deflection described by an incomplete (nine-term) cubic polynomial	[9]
QUAD4		4	<i>T, R</i>	Isoparametric with selective reduced order integration. Transverse shear uses string-net approximation and augmented shear flexibility	[10,11]
QUAD8		8	<i>T, R</i>	Isoparametric with selective reduced order integration	[12]
HEXA(8)		8	<i>T</i>	Isoparametric with selective reduced order integration and internal strain functions equivalent to bubble functions	[13,14]
HEX20		20	<i>T</i>	Standard isoparametric	[15]
HEX20(R)		20	<i>T</i>	Standard isoparametric with reduced order integration.	[16]

^a Note: *T* = all three components of translation; *R* = all three components of rotation.

The section of a thick-walled cylinder shown in Fig. 10 has been chosen to test the effect of nearly incompressible material. Note that a plane strain condition is assumed which, along with the radial symmetry, confines the material in all but the radial direction and intensifies the numerical difficulty caused by near incompressibility.

The elements

The ability of the proposed test problems to discriminate good and poor element accuracy will be examined by testing a selection of NASTRAN[®] and MSC/NASTRAN elements. The selection includes three quadrilateral shell elements (QUAD2, QUAD4, and QUAD8) and three solid brick elements (HEXA(8), HEX20, and HEX20(*R*)). The properties of each of the six elements² are summarized in Table 6.

The QUAD2 element is the original NASTRAN[®] quadrilateral plate element. It was conceived in 1967 and it is still heavily used in NASTRAN[®]. It has been replaced in MSC/NASTRAN by the QUAD4, which is probably the most frequently used MSC/NASTRAN element, and by the eight noded QUAD8.

² One of the authors of this paper, R.H. MacNeal, is also the author of four of the elements (QUAD2, QUAD4, QUAD8, and HEXA(8)) and therefore need not apologize to anyone, except perhaps to the users of these elements, for any unkind words that may be said here about them.

The MSC/NASTRAN brick element, HEXA, has both an eight-noded and a twenty-noded form. Only the more frequently used eight-noded form, HEXA(8), will be tested. The other two brick elements to be tested are the infrequently used MSC/NASTRAN HEX20 and HEX20(R), which are included here because they are respectively identical to the well known and widely used standard isoparametric brick element with full and reduced order integration.

The QUAD4, QUAD8, and HEXA(8) elements are modified occasionally, and some of the modifications may affect accuracy. The results to be presented were obtained with Version 63 of MSC/NASTRAN and may not agree in all details with results from earlier or later versions.

Test results

The test results are detailed in Tables 7 through 15 and summarized in Tables 16 and 17. The test data have been restricted to one data point per case in order to save space. The quantity recorded in the patch test results (Table 7) is the maximum stress error. In cases with point loads, the recorded quantity is the displacement under load, which is a direct measure of the total strain energy. In cases with distributed loads, the largest displacement component has been recorded.

Most of the data in Tables 7 through 15 are presented in normalized form, i.e., the recorded value is the ratio of the value obtained in an MSC/NASTRAN finite element analysis to the theoretical value. The one exception occurs in the patch test where the percent error is recorded.

In Tables 16 and 17, the test data are reduced to a letter grade by the following rules:

Grade	Rule
A	2% \geq error
B	10% \geq error > 2%
C	20% \geq error > 10%
D	50% \geq error > 20%
F	error > 50%

In situations where more than one case contributes to a letter grade, the absolute errors have been averaged before assigning a letter grade.

Tables 16 and 17 reveal a great deal about the elements and about the tests. It should be noted, first of all, that the proposed test problems are deceptively difficult and that a failing grade in one or more problems should not disqualify the element for applications where the combination of parameters that caused the failure do not occur. Note, for example, that both the QUAD2 and the QUAD8 fail the patch test for out-of-plane loading. The QUAD2 then proceeds to fail four of the eight other tests that involve out-of-plane loading while the QUAD8 fails none. The test scores of all the elements could, of course, be improved by mesh refinement, particularly in the beam problems where the mesh is only a single element deep in the slender directions. On the other hand, most users expect to get accurate results with coarse meshes, and the authors have constructed the test problems in this spirit.

Table 7
Patch test results

	Maximum error in stress					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
Constant-stress loading	0.0	0.0	18.0%	0.0	0.0	0.0
Constant-curvature loading	30.7%	0.0	51.6%	N/A	N/A	N/A

The most disturbing failure of the QUAD2 element is its inability to get a passing grade for in-plane bending of the straight beam with regular element shape. This failure is echoed in the curved beam and the Scordelis-Lo roof problems and was responsible for the development of the QDMEM1 and QDMEM2 membrane elements in NASTRAN[®], neither of which proved to be a significant improvement. The spectacular failure of the QUAD2 in the twisted beam problem is due to the misalignment of the moments at the boundaries between adjacent elements when the elements are skewed. The misalignment produces a resultant moment about the normal to the surface which is not resisted. In spite of these weaknesses, the QUAD2 received an excellent grade on the spherical shell problem in contrast to the higher order elements which received C's or worse.

The only failures of the QUAD4 element are locking for the straight beam with in-plane

Table 8
Results for straight cantilever beam

Tip loading direction	Normalized tip displacement in direction of load					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R) ^a
<i>(a) Rectangular elements</i>						
Extension	0.992	0.995	0.999	0.988	0.994	0.999
In-plane shear	0.032	0.904	0.987	0.981	0.970	0.984
Out-of-plane shear	0.971	0.986	0.991	0.981	0.961	0.972
Twist	0.566	0.941	0.950	0.910	0.904	0.911
<i>(b) Trapezoidal elements</i>						
Extension	0.992	0.996	0.999	0.989	0.994	0.999
In-plane shear	0.016	0.071	0.946	0.069	0.886	0.964
Out-of-plane shear	0.963	0.968	0.998	0.051	0.920	0.964
Twist	0.616	0.951	0.943	0.906	0.904	0.918
<i>(c) Parallelogram elements</i>						
Extension	0.992	0.996	0.999	0.989	0.994	0.999
In-plane shear	0.014	0.080	0.995	0.080	0.967	0.994
Out-of-plane shear	0.961	0.977	0.985	0.055	0.941	0.961
Twist	0.615	0.945	0.965	0.910	0.904	0.913

^a The good to excellent results shown here for HEX20(R) were obtained in spite of singularity in the beam's stiffness matrix.

Table 9
Results for curved beam

Tip loading direction	Normalized tip displacement in direction of load					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
In-plane (vertical)	0.825	0.833	1.007	0.880	0.875	1.006
Out-of-plane	0.594	0.951	0.971	0.849	0.946	0.959

Table 10
Results for twisted beam

Tip loading direction	Normalized tip displacement in direction of load					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
In-plane	100.4	0.993	0.998	0.983	0.991	0.993
Out-of-plane	228.9	0.985	0.998	0.977	0.995	0.999

Table 11
Results for rectangular plate simple supports: uniform load

(a) Aspect ratio = 1.0						
Number of node spaces ^a per edge of model	Normalized lateral deflection at center					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.971	0.981	0.927	0.989	0.023	1.073
4	0.995	1.004	0.996	0.998	0.738	0.993
6	0.998	1.003	0.999	0.999	0.967	1.011
8	0.999	1.002	1.000	1.000	0.991	1.008

(b) Aspect ratio = 5.0						
Number of node spaces ^a per edge of model	Normalized lateral deflection at center					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.773	1.052	1.223	0.955	0.028	1.139
4	0.968	0.991	1.003	0.978	0.693	0.995
6	0.993	0.997	1.000	0.990	1.066	1.024
8	0.998	0.998	1.000	0.995	1.026	1.006

^a For elements with midside nodes, the number of elements per edge of model is equal to one-half the number of node spaces.

Table 12
Results of rectangular plate clamped supports: concentrated load

(a) Aspect ratio = 1.0						
Number of node spaces per edge of model	Normalized lateral deflection at center					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.979	0.934	1.076	0.885	0.002	0.983
4	1.008	1.010	0.969	0.972	0.072	0.433
6	1.006	1.012	0.992	0.988	0.552	0.813
8	1.005	1.010	0.997	0.994	0.821	0.942

(b) Aspect ratio = 5.0						
Number of node spaces per edge of model	Normalized lateral deflection at center					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.333	0.519	0.542	0.321	0.001	0.363
4	0.512	0.863	0.754	0.850	0.041	0.447
6	0.638	0.940	0.932	0.927	0.220	0.721
8	0.723	0.972	0.975	0.957	0.374	0.867

Table 13
Results for Scordelis–Lo roof

Number of node spaces per edge of model	Normalized vertical deflection at midpoint of free edge					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.784	1.376	1.021	1.320	0.092	1.046
4	0.665	1.050	0.984	1.028	0.258	0.967
6	0.781	1.018	1.002	1.012	0.589	1.003
8	0.854	1.008	0.997	1.005	0.812	0.999
10	0.897	1.004	0.996	–	–	–

bending and irregular elements and locking for the thick-walled cylinder with nearly incompressible material. These are state-of-the-art problems for four-noded elements. The only other test where QUAD4 shows weakness is the curved beam with in-plane loading which is a manifestation of the first mentioned locking problem.

The QUAD8 receives passing grades on all but the out-of-plane patch test, which is surprising in view of the importance usually attached to the patch test. On the other hand, the C

Table 14
Results for spherical shell

Number of node spaces per edge of model	Normalized radial deflection at load point					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
2	0.928	0.972	0.025	—	—	—
4	0.990	1.024	0.121	0.039	0.001	0.162
6	0.990	1.013	0.494	—	—	—
8	0.986	1.005	0.823	0.730	0.021	0.776
10	0.984	1.001	0.955	—	—	—
12	0.982	0.998	0.992	0.955	0.097	0.972

Table 15
Results for thick-walled cylinder

Poisson's ratio	Normalized radial displacement at inner boundary					
	QUAD2	QUAD4	QUAD8	HEXA(8)	HEX20	HEX20(R)
0.49	0.643	0.846	1.000	0.986	0.999	1.000
0.499	0.156	0.359	0.997	0.986	0.986	1.000
0.4999	0.018	0.053	0.967	0.986	0.879	1.000

Table 16
Summary of test results for shell elements

Test	Element loading		Element shape ^a	QUAD2	QUAD4	QUAD8
	In-plane	Out-of-plane				
(1) Patch test	×		Irregular	A	A	C
(2) Patch test		×	Irregular	D	A	D
(3) Straight beam, extension	×		All	A	A	A
(4) Straight beam, bending	×		Regular	F	B	A
(5) Straight beam, bending	×		Irregular	F	F	B
(6) Straight beam, bending		×	Regular	B	A	A
(7) Straight beam, bending		×	Irregular	B	B	A
(8) Straight beam, twist			All	D	B	B
(9) Curved beam	×		Regular	F	C	A
(10) Curved beam		×	Regular	D	B	B
(11) Twisted beam	×	×	Regular	F	A	A
(12) Rectangular plate ($N = 4$)		×	Regular	C	B	B
(13) Scordelis–Lo roof ($N = 4$)	×	×	Regular	D	B	A
(14) Spherical shell ($N = 8$)	×	×	Regular	A	A	C
(15) Thick-walled cylinder ($\nu = 0.4999$)	×		Regular	F	F	B
Number of failed tests (D's and F's)				9	2	1

^a Regular means that element shape has not been intentionally distorted.

noted for the spherical shell is disturbing in view of the practical importance of this problem.

The HEXA(8) element passes the patch test and performs well when the elements are regular but behaves poorly for bending loads on irregular elements. This is probably why it also converges slowly in the spherical shell problem and gets C's in the curved beam test where the elements deviate slightly from rectangular shape. It is noteworthy that the HEXA(8) gets an excellent mark for the nearly incompressible thick-walled cylinder as the result of a design feature recently built into the element.

The standard isoparametric HEX20 does fairly well on all the beam problems but fails the important plate and shell problems. These failures are somewhat alleviated by the introduction of reduced order integration as evidenced in the results for HEX20(*R*). On the other hand, reduced order integration is responsible for the singularity of the stiffness matrix noted in the straight beam tests. The spurious modes associated with this singularity involve a second order distortion of the cross-section which has no component in the direction of the applied load. The rather erratic results for HEX20(*R*) shown in Table 12(a) are probably related to the reduced order of integration. Thus, while reduced order integration appears to improve performance for many applications, it can lead to sudden unexpected failure in others.

Turning our attention away from the elements to the test problems, it is seen that the problems are capable of evoking both good and bad results. In only one problem, extension of a straight beam, did all of the elements get A's. (This is a throwaway test in the sense that any element that cannot pass it should be thrown away.) All but three of the remaining fourteen tests produced failing grades for one or more elements and all but three of the remaining tests produced a grade of A for one or more elements.

Two important questions are whether any of the tests are duplicates of other tests and whether the tests are sufficient. The first question can be answered by noting that the letter grade test results are not duplicated across the six elements by any pair of tests. The second question is more difficult to answer because the list of error causing parameters is not closed. We can, however, examine whether the parameters described in the introduction as those which affect accuracy are present in one or more problems. The result of the examination is

Table 17
Summary of test results for solid elements

Test	Element shape	HEXA(8)	HEX20	HEX20(<i>R</i>)
(1,2) Patch test	Irregular	A	A	A
(3) Straight beam, extension	All	A	A	A ^c
(4,6) Straight beam, bending	Regular	A	B	B ^c
(5) Straight beam, bending	Irregular ^a	F	B	A ^c
(7) Straight beam, bending	Irregular ^b	F	B	B ^c
(8) Straight beam, twist	All	B	B	B ^c
(9) Curved beam, in-plane loading	Regular	C	C	A
(10) Curved beam, out-of-plane loading	Regular	C	B	B
(11) Twisted beam	Regular	B	B	A
(12) Rectangular plate ($N = 4$)	Regular	B	F	D
(13) Scordelis-Lo roof ($N = 4$)	Regular	B	F	B
(14) Spherical shell ($N = 8$)	Regular	D	F	D
(15) Thick-walled cylinder ($\nu = 0.4999$)	Regular	A	C	A
Number of failed tests (D's and F's)		3	3	2

^a Bending in plane of irregularity.

^b Bending out of plane of irregularity.

^c In spite of singular stiffness matrix.

Table 18
Accuracy affecting parameters vs. test problems

Parameters which affect accuracy	Test problems where illustrated
<i>Response excited by loading</i>	
Extension	1, 3, 8, 9, 10, 13, 14, 15
In-plane shear and bending	1, 4, 5, 9, 11, 13, 14, 15
Out-of-plane shear and bending	2, 6, 7, 10, 11, 12, 13, 14
Twist	2, 8, 10, 12, 13, 14
Higher order stress gradients	12, 13, 14, 15
<i>Element shape</i>	
Aspect ratio	All
Skew	1, 2, 3, 5, 7, 8
Taper	1, 2, 3, 5, 7, 8, 9, 10, 13, 14, 15
Warp	11
Displaced edge nodes	9, 10, 13, 14, 15
<i>Problem geometry</i>	
Single curvature	13
Double curvature	11, 14
Free boundary	3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15
Simply-supported boundary	12, 13
Clamped boundary	3, 4, 5, 6, 7, 8, 9, 10, 11, 12
Symmetric boundary	12, 13, 14, 15
Enforced motion and boundary	1, 2
Thinness	All except 15
Slenderness	3, 4, 5, 6, 7, 8, 9, 10, 11
<i>Material</i>	
Nearly incompressible	15
Anisotropic	None

summarized in Table 18, where it is seen that all of the parameters affecting accuracy except anisotropic material are present in one or more of the proposed test problems and more often in several of them.

Concluding remarks

The present paper has described a proposed standard set of finite element test problems in sufficient detail to allow their reproduction in any general purpose finite element code. The problems, while few in number and relatively simple, have been designed to include, collectively, nearly all of the parameters which have important effects on element accuracy.

The proposed test problems have been exercised on a representative set of quadrilateral shell and brick elements with different orders of complexity, and it has been shown that almost every problem is capable of evoking results ranging from excellent to poor. As developers of finite elements, this is exactly the sort of test problem library we would want to help us find and correct element errors and weaknesses. It is also hoped that the user community will be able to profit from them in selecting elements and mesh sizes.

Acknowledgment

The work reported here has been prepared in response to a request from the AIAA Standards Committee on Structural Analysis, chaired by Kevin J. Forsberg.

References

- [1] ROBINSON, J., and S. BLACKHAM, "An evaluation of lower order membranes as contained in the MSC/NASTRAN, ASAS, and PAFEC FEM systems", Robinson and Associates, Dorset, England, 1979.
- [2] ROBINSON, J., and S. BLACKHAM, "An evaluation of plate bending elements: MSC/NASTRAN, ASAS, PAFEC, ANSYS, and SAP4", Robinson and Associates, Dorset, England, 1981.
- [3] *The NASTRAN Theoretical Manual*, level 16.0, NASA SP-221(03), Section 15.2, 1976.
- [4] IRONS, B. and S. AHMAD, *Techniques of Finite Elements*, Wiley, New York, p. 155, 1980.
- [5] SCORDELIS, A.C. and K.S. LO, "Computer analysis of cylindrical shells", *J. Amer. Concr. Inst.* **61**, pp. 539–561, 1969.
- [6] ZIENKIEWICZ, O.C., *The Finite Element Method*, McGraw-Hill, London, p. 417, 1977.
- [7] MORLEY, L.S.D. and A.J. MORRIS, *Conflict Between Finite Elements and Shell Theory*, Royal Aircraft Establishment Report, London, 1978.
- [8] TIMOSHENKO, S., and S. WOINOWSKY-KRIEGER, *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, pp. 120, 143, 202, 206, 1959.
- [9] *The NASTRAN Theoretical Manual*, Level 16.0, NASA SP-221(03), Section 5.8.3, 1976.
- [10] MACNEAL, R.H., "A simple quadrilateral shell element", *Computers and Structures* **8**, pp. 175–183, 1978.
- [11] MACNEAL, R.H., "Derivation of element stiffness matrices by assumed strain distributions", *Nuclear Engineering and Design* **70**, pp. 3–12, 1982.
- [12] MACNEAL, R.H., "Specifications for the QUAD8 quadrilateral curved shell element," MacNeal-Schwendler Corp. Memo. RHM-46B, 1980.
- [13] MACNEAL, R.H., "Specifications for the HEXA element", MacNeal-Schwendler Corp. Memo. RHM-38C, 1976.
- [14] HARDER, R.L., "Modified isoparametric elements and patch tests", MacNeal-Schwendler Corp. Memo. RLH-35, 1982.
- [15] IRONS, B.M., "Engineering application of numerical integration in stiffness method", *J. AIAA* **14**, pp. 2035–2037, 1966.
- [16] ZIENKIEWICZ, O.C., R.L. TAYLOR and J.M. TOO, "Reduced integration techniques in general analysis of plates and shells", *Internat. J. Numer. Method. Engrg.* **3**, pp. 275–290, 1971.